A SIMPLE PROOF AND SOME DIFFICULT EXAMPLES FOR HINDMAN'S THEOREM

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ABSTRACT. We give a short, explicit proof of Hindman's Theorem that in every finite coloring of the integers, there is an infinite set all of whose finite sums have the same color. We give several exampls of colorings of the integers which do not have computable witnesses to Hindman's Theorem; in particular, we improve the recursion-theoretic lower bound by giving a coloring with no Σ_2 monochromatic IP set.

1. INTRODUCTION

Hindman's Theorem is:

Theorem 1.1. *If $c : \mathbb{N} \to [1, r]$ is given then there are an $i \in [1, r]$ and an an infinite set S such that $c(s) = i$ whenever s is the sum of one or more distinct elements of S.*

There are three standard proofs of Hindman's theorem: the original combinatorial argument ([4]), a streamlined combinatorial argument ([1]), and the Galvin-Glazer proof using ultrafilters (see [3] or [6]). The original proof is generally considered quite difficult (see, for instance, the comments on it in [5]), but work in reverse mathematics shows that it is also, at least in the sense of reverse mathematics, the simplest of the three proofs. Specifically, Blass, Hirst, and Simpson have shown ([2]) that Hindman's proof can be formalized in the system $\mathbf{ACA_0^+}$, while Baumgartner's proof requires the stronger system $\mathbf{\Pi^1_2} - \mathbf{TI_0}$. The Galvin-Glazer proof was analyzed in ([8]), where it was shown that it requires an even stronger system than Baumgartner's. (The definitions and significance of all these systems of reverse mathematics may be found in ([7]).)

The work in [8] demonstrated a striking analogy between the structures of Baumgartner and Galvin-Glazer proofs: roughly speaking, both proofs prove an intermediate theorem that a structure weaker than that promised by Hindman's Theorem exists, then repeat the same argument with one step replaced by the intermediate theorem. Hindman's proof does not have this structure, but comparison of the proofs suggests that the corresponding intermediate would be the structure given by Theorem 2.5 below. With the use of this intermediate, we can give a new proof similar to Hindman's (and provable, like Hindman's, in $\mathbf{ACA_0^+}$).

[2] also gives a lower bound for the reverse mathematical strength of Hindman's Theorem by constructing a computable coloring such that $\mathbf{0}'$ is computable in any set witnessing Hindman's Theorem. In particular, Hindman's Theorem implies $\mathbf{ACA_0}$ over $\mathbf{RCA_0}$. We give a small improvement in the recursion theoretic strength of this lower bound by constructing a coloring such with no Σ_2 set witnessing Hindman's Theorem. We also give additional colorings showing that certain aspects of our proof of Hindman's Theorem are optimal.

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2. A Simple Proof of Hindman's Theorem

It is standard (see [1]) to take advantage of the fact that Hindman's Theorem is equivalent to a similar statement about unions of finite sets. We will freely equate $\mathcal{P}_{fin}(\mathbb{N})$ with $\mathbb{N}$, using the fact that there is a computable bijection between the two sets.

Definition 2.1. *If $\mathcal{S} \subseteq \mathcal{P}_{fin}(\mathbb{N})$, we write $NU(\mathcal{S})$ for the set of* non-empty unions *from $\mathcal{S}$, those non-empty T which are the union of finitely many element of $\mathcal{S}$.*

We say $\mathcal{S} \subseteq \mathcal{P}_{fin}(\mathbb{N})$ is IP if it is closed under finite unions and contains an infinite set of pairwise disjoint elements.

If $B \in \mathcal{S}$, we will write

$$\mathcal{S} - B := \{T \in \mathcal{S} \mid T \cap B = \emptyset\},$$

and if $\mathcal{B} \subseteq \mathcal{S}$ then

$$\mathcal{S} - \mathcal{B} := \bigcap_{B \in \mathcal{B}} \mathcal{S} - B.$$

Then subtraction is a strong form of set difference, where we remove not only B, but also anything that intersects B.

The following theorem is easily seen to imply Hindman's Theorem (consider the map taking a number n to the set of places which are 1 in the binary expansion of n). (With more work, it can be seen to follow from Hindman's Theorem as well.)

Theorem 2.2 (Finite Unions Theorem)**.** *If $c : \mathcal{P}_{fin}(\mathbb{N}) \to [1, r]$ is given then there are an $i \in [1, r]$ and an IP set $\mathcal{S}$ such that $c(S) = i$ for every $S \in \mathcal{S}$.*

We introduce two weak notions which will characterize our intermediate steps:

Definition 2.3. *We say $\mathcal{D}$* half-matches *B if there is a $D \in \mathcal{D}$ such that $c(B) = c(D \cup B)$. We say $\mathcal{D}$ half-matches a set $\mathcal{B}$ if $\mathcal{D}$ half-matches every $B \in \mathcal{B}$.*

We say $\mathcal{D}$ full-matches *B if there is a $D \in \mathcal{D}$ such that $c(D) = c(B) = c(D \cup B)$. We say $\mathcal{D}$ full-matches a set $\mathcal{B}$ if $\mathcal{D}$ full-matches every $B \in \mathcal{B}$.*

Lemma 2.4 ($\mathbf{RCA_0}$)**.** *Let $\mathcal{S}$ be an IP set, let $\mathcal{B} \subseteq \mathcal{S}$ be finite, and let $c : NU(\mathcal{S}) \to [1, r]$ be given. Then either:*

- *There is a finite $\mathcal{D} \subseteq \mathcal{S} - \mathcal{B}$ such that for every $S \in \mathcal{S} - \mathcal{B} - \mathcal{D}$, there is a $D \in NU(\mathcal{D})$ such $\mathcal{B}$ does not half-match $D \cup S$, or*
- *There is an IP set $\mathcal{T} \subseteq \mathcal{S} - \mathcal{B}$ such that $\mathcal{B}$ half-matches $\mathcal{T}$.*

Proof. Suppose the first condition fails; that is, for any finite $\mathcal{D} \subseteq \mathcal{S} - \mathcal{B}$, there is an $S \in \mathcal{S} - \mathcal{B} - \mathcal{D}$ such that $\mathcal{B}$ half-matches $D \cup S$ for every $D \in \mathcal{D}$.

We inductively construct a sequence $\mathcal{D}_0 \subseteq \mathcal{D}_1 \subseteq \cdots$ of finite subsets of $\mathcal{S} - \mathcal{B}$ such that whenever $D \in NU(\mathcal{D}_n) \setminus \mathcal{D}_n$, $\mathcal{B}$ half-matches D. Set $\mathcal{D}_0 := \{D_0\}$ for an arbitrary $D_1 \in \mathcal{S} - \mathcal{B}$. Given $\mathcal{D}_n$, since the first condition fails and $NU(\mathcal{D}_n)$ is finite, there is an $S \in \mathcal{S} - \mathcal{B} - \mathcal{D}_n$ such that for every $D \in NU(\mathcal{D}_n)$, $\mathcal{B}$ half-matches $D \cup S$. Let $\mathcal{D}_{n+1} := \mathcal{D}_n \cup \{S\}$. Then for any $D \in NU(\mathcal{D}_{n+1}) \setminus \mathcal{D}_{n+1}$, either $D \in NU(\mathcal{D}_n) \setminus \mathcal{D}_n$, in which case $\mathcal{B}$ half-matches D by IH, or $D = D' \cup S$ for some $D' \in \mathcal{D}_n$, in which case $\mathcal{B}$ half-matches D by choice of S.

Let $\mathcal{D} := \bigcup_n \mathcal{D}_n = \{D_0, D_1, \ldots\}$. Let $\mathcal{D}' := \{D_{2i} \cup D_{2i+1} \mid i \in \mathbb{N}\}$. Then if $D \in NU(\mathcal{D}')$, $D \in NU(\mathcal{D}_n) \setminus \mathcal{D}_n$ for some n, so $\mathcal{B}$ half-matches D. □

Lemma 2.5 ($\mathbf{RCA_0}$). *If $\mathcal{S}$ is an IP set and $c : NU(\mathcal{S}) \to [1, r]$ then there is a finite collection $\mathcal{B} \subseteq \mathcal{S}$ and an IP set $\mathcal{T} \subseteq \mathcal{S} - \mathcal{B}$ such that $\mathcal{B}$ half-matches $\mathcal{T}$.*

Proof. Pick an arbitrary element $Q \in \mathcal{S}$, and set $\mathcal{B}_1 := \{Q\}$ and $\mathcal{S}'_1 := \mathcal{S} - \mathcal{B}_1$. Given $\mathcal{B}_i, \mathcal{S}'_i$, apply Lemma 2.4. If the second condition holds, we are finished. Otherwise let $\mathcal{D}_{i+1}$ be given by the first part, let $\mathcal{B}_{i+1} := NU(\mathcal{B}_i \cup \mathcal{D}_{i+1})$, and let $\mathcal{S}'_{i+1} := \mathcal{S}'_i - \mathcal{B}_{i+1}$.

Suppose that we reach $\mathcal{B}_r, \mathcal{S}'_r$ without terminating. Then for any $S \in \mathcal{S}'_r$, we may choose a sequence $D_r, \ldots, D_2$ with $D_i \in NU(\mathcal{D}_i)$ and for each i, $\mathcal{B}_{i-1}$ fails to half-match $S \cup \bigcup_{j=i}^{r} D_j$. Let $D_1 := Q$. Then for each $i < i'$, since $\bigcup_{j=i}^{i'-1} D_j \in \mathcal{B}_{i'-1}$, $c(S \cup \bigcup_{j=i}^{r} D_j) \neq c(S \cup \bigcup_{j=i'}^{r} D_j)$. But since there are r colors, there must be some i such that $c(S) = c(S \cup \bigcup_{j=i}^{r} D_j)$. Therefore we may take $\mathcal{B} := \mathcal{B}_r$ and $\mathcal{T} := \mathcal{S}'_r$. $\square$

Lemma 2.6 ($\mathbf{ACA_0}$). *Let $\mathcal{S}$ be an IP set and let $c : \mathcal{S} \to [1, r]$ be given. Then either:*

- *There is an IP $\mathcal{S}' \subseteq \mathcal{S}$ and some $i \in [1, r]$ such that $c(S) \neq i$ for every $S \in \mathcal{S}'$, or*
- *There is a finite collection $\mathcal{B} \subseteq \mathcal{S}$ and an IP set $\mathcal{T} \subseteq \mathcal{S} - \mathcal{B}$ such that $\mathcal{B}$ full-matches $\mathcal{T}$.*

Proof. Construct sequences $\mathcal{B}_2, \ldots, \mathcal{B}_n, \ldots$, $\mathcal{T}_1, \ldots, \mathcal{T}_n, \ldots$, and colorings $c_1, \ldots, c_n, \ldots$ as follows: let $c_1 := c$ and $\mathcal{T}_1 := \mathcal{S}$. Given $c_i, \mathcal{T}_i$, let $\mathcal{B}_{i+1}, \mathcal{T}_{i+1}$ be the witness given by Lemma 2.5. Define c_{i+1} on $\mathcal{T}_{i+1}$ by setting $c_{i+1}(S) := \langle B, c_i(S) \rangle$ where $B \in \mathcal{B}_{i+1}$ is such that $c_i(S) = c_i(S \cup B)$.

If there is some n such that for every $S \in \mathcal{T}_n$ there is a $B \in NU(\bigcup_{i \leq n} \mathcal{B}_i)$ such that $c(S) = c(B) = c(S \cup B)$ then $\mathcal{T}_n$ and $\bigcup_{i \leq n} \mathcal{B}_i$ witness the second possibility.

Otherwise, for each n we may choose a $T_n \in \mathcal{T}_n$ such that there is no $B \in NU(\bigcup_{i \leq n} \mathcal{B}_i)$. By the pigeonhole principle, we may choose an infinite subsequence $\{T_{i_n}\}$ such that c is constantly some fixed $q \in [1, r]$ on $\{T_{i_n}\}$ (but not necessarily on $NU(\{T_{i_n}\})$). For each T_{i_n}, we may choose a sequence $B_1 \in \mathcal{B}_1, \ldots, B_{i_n} \in \mathcal{B}_{i_n}$ such that $c(T_{i_n}) = c(T_{i_n} \cup B)$ for every $B \in NU(\{B_i\})$. In particular, it must be that $c(B) \neq q$.

Then by König's Lemma, we may choose an infinite sequence $\{B_i\}$ such that $c(B) \neq q$ for any $B \in NU(\{B_1, \ldots, B_n, \ldots\})$. $\square$

Note that, when the second clause holds in the preceeding lemma, the set $\mathcal{T}$ is computable from c and $\mathcal{S}$.

Lemma 2.7 ($\mathbf{ACA_0}$). *Let $\mathcal{S}$ be an IP set and let $c : \mathcal{S} \to [1, r]$ be given. Then either:*

- *There is an IP $\mathcal{S}' \subseteq \mathcal{S}$ such that c is constant on $\mathcal{S}'$, or*
- *There is a finite collection $\mathcal{B} \subseteq \mathcal{S}$ and an IP set $\mathcal{T} \subseteq \mathcal{S} - \mathcal{B}$ such that $\mathcal{B}$ full-matches $\mathcal{T}$.*

Proof. By induction on r. When $r = 1$, the first condition holds immediately. If the claim holds for r and $c : NU(\mathcal{S}) \to [1, r+1]$, we may apply Lemma 2.6 and either reduce to IH or immediately give the second case. $\square$

Theorem 2.8 ($\mathbf{ACA_0^+}$). *If $c : \mathcal{P}_{fin}(\mathbb{N}) \to [1, r]$ is given then there are an $i \in [1, r]$ and an IP set $\mathcal{S}$ such that $c(S) = i$ for every $S \in \mathcal{S}$.*

Proof. The method is the same as Lemma 2.6. Construct sequences $\mathcal{B}_2, \ldots, \mathcal{B}_n, \ldots$, $\mathcal{T}_1, \ldots, \mathcal{T}_n, \ldots$, and colors $c_1, \ldots, c_n, \ldots$ by setting $c_1 := c$ and $\mathcal{T}_1 := \mathcal{S}$. Given $c_i, \mathcal{T}_i$, apply Lemma 2.7; in the first case, we are done. In the second, let $\mathcal{B}_{i+1}, \mathcal{T}_{i+1}$ be the given witness and define c_{i+1} on $NU(\mathcal{T}_{i+1})$ by setting $c_{i+1}(S) := \langle B, c_i(S) \rangle$ where $B \in \mathcal{B}_{i+1}$ is such that $c_i(S) = c_i(B) = c_i(S \cup B)$.

Then for any n, we may find a sequence $\{B_i\}_{i\leq n}$ with $B_i \in \mathcal{B}_i$ and c constant on $NU(\{B_i\}_{i\leq n})$. By Weak König's Lemma, we may find an infinite sequence $\{B_i\}$ so that c is constant on $NU(\{B_i\})$, as promised. □

3. Difficult Examples

In [2], a lower bound for the reverse mathematical strength of Hindman's Theorem is established by exhibiting a computable coloring of $\mathcal{P}_{fin}(\mathbb{N})$ which has no computable monochromatic IP set. Specifically, two such colorings are given, one where every monochromatic IP set computes $\mathbf{0}'$ and one where no monochromatic IP set is computable in $\mathbf{0}'$.

In this section, we present computable colorings of $\mathcal{P}_{fin}(\mathbb{N})$ with various more specific properties. We hope to serve three purposes: First, we will improve the recursion theoretic lower bound on Hindman's Theorem by giving a computable coloring of $\mathcal{P}_{fin}(\mathbb{N})$ with no Σ_2 monochromatic IP set. Second, we will demonstrate that various stages in the proof of the previous section are optimal; if one hopes to give a proof of Hindman's Theorem within $\mathbf{ACA_0}$, this will help indicate where improvements are possible. Finally, since these are the first new examples of colorings which are computationally difficult for Hindman's Theorem, we hope the relatively flexible nature of our method will spur the development of further progress.

We adopt a few notational conventions. Whenever we write the union of two finite sets, say $B \cup C$, we always assume that $\max B < \min C$. We say $\mathcal{S}$ *generates* an IP set if $\mathcal{S}$ contains infinitely many pairwise disjoint elements. (That is, $\mathcal{S}$ generates an IP set iff $NU(\mathcal{S})$ is an IP set.) When we speak of one set B containing a set C, we mean that $B = A_0 \cup C \cup A_1$ with $\max A_0 < \min C$, $\max C < \min A_0$ (and possibly A_0, A_1 or both empty). We fix some ordering $\prec$ of $\mathcal{P}_{fin}(\mathbb{N})$ with order type ω so that if $\min B < \min C$ then $B \prec C$.

We will let $\mathcal{W}_1, \ldots, \mathcal{W}_i, \ldots$ be an enumeration of the computably enumerable subsets of $\mathcal{P}_{fin}(\mathbb{N})$, and for each i, s, define $\mathcal{W}_{i,s}$ to be a finite subset of $\mathcal{P}_{fin}(\mathbb{N})$ computable from i, s such that $s \leq t$ implies $\mathcal{W}_{i,s} \subseteq \mathcal{W}_{i,t}$ and $\mathcal{W}_i = \bigcup_s \mathcal{W}_{i,s}$.

Before giving examples, we briefly describe our method, which is modeled on the finite injury priority argument. (This idea was suggested to us by Carl Mummert.) We will fix a list of conditions, indexed by the natural numbers, which we wish our coloring to satisfy; for instance, we might want to ensure that each of the countably many computably enumerable sets either fails to generate an IP set or generates one which is not monochromatic.

In this case, the i-th condition wishes to choose two elements of $\mathcal{W}_i$ and color them distinct colors. However, since $\mathcal{W}_i$ is only computably enumerable, and we want our coloring to be computable, we must decide how to color a given set without being able to wait to see whether it will be in $\mathcal{W}_i$. Instead, we will wait until some $W \in \mathcal{W}_{i,s}$ for some big enough s, and then color sets of the form $W \cup B$ where $\max B \geq s$. If $\mathcal{W}_i$ generates an IP set, we are guaranteed that we can find a $B \in \mathcal{W}_i$ with $\max B \geq s$ (and $\max W < \min B$), and we will therefore have $W \cup B \in NU(\mathcal{W}_i)$ be an element colored according to our desired rule.

3.1. A Computable Coloring with No Computably Enumerable Monochromatic IP Set.

To illustrate our method, we give a coloring with no computably enumerable monochromatic IP set. Our method is similar to (though gives a weaker result than) Theorem 2.1 of [2].

Theorem 3.1. *There is a computable $c : \mathcal{P}_{fin}(\mathbb{N}) \to \{0,1\}$ such that if $\mathcal{S}$ is computably enumerable and generates an IP set then $FU(S)$ is not monochromatic.*

Proof. For any s and each $i \leq s$, we define W_i^s to be least (with respect to $\prec$) such that:

- $W_i^s \in \mathcal{W}_{\lfloor i/2 \rfloor, s}$
- If $j < i$ and W_j^s is defined then $\max W_j^s < \min W_i^s$

If there is no such element then W_i^s is undefined.

Given $B \in \mathcal{P}_{fin}(\mathbb{N})$ with $\max B = s$, observe that there is at most one i such that W_i^s is an initial segment of B. If there is no such i, set $c(B) = 0$. If there is such an i, set $c(B) = i \mod 2$.

Claim 1. *For each i, there is some s such that $W_i^s = W_i^t$ for all $t \geq s$ (where both sides are undefined if either is).*

Proof. By induction on i. Let s_0 be large enough such that for all $j < i$, if $t \geq s_0$ then $W_j^{s_0} = W_j^t$. If $\mathcal{W}_{\lfloor i/2 \rfloor}$ contains any W such that $\min W > \max W_j^{s_0}$ for all $j < i$, there is a least such W. There must be some s such that $W \in W_{\lfloor i/2 \rfloor, s}$, and it follows that $W_i^t = W$ for all $t \geq \max s, s_0$. Otherwise, there is no such W, so W_i^t is undefined for all $t \geq s_0$. $\dashv$

Then the following follows immediately:

Claim 2. *If $\mathcal{W}_e$ generates an IP set then there is some s such that for all $t \geq s$, W_{2e}^t, W_{2e+1}^t are defined.*

Suppose $\mathcal{W}_e$ generates an IP set. Then in particular, it contains some W_0, W_1 with $\max W_0 < \min W_1$ such that for some s, $W_{2e}^t = W_0$ and $W_{2e+1}^t = W_1$ for all $t \geq s$. Since $\mathcal{W}_e$ contains infinitely many pairwise disjoint elements, it must contain some B with $\max B \geq s$ and $\min B > \max W_1$. It follows that $c(W_0 \cup B) = 0$ and $c(W_1 \cup B) = 1$. Since $W_0 \cup B, W_1 \cup B \in FS(W_e)$, it follows that W_e does not generate a monochromatic IP set. $\square$

3.2. **Computable Colorings with No Computably Enumerable Sets Half-Matched by Small Sets.** Here we show that there is no bound on the size of the finite set $\mathcal{B}$ found in Lemma 2.5.

Theorem 3.2. *For any k, there is a computable $c : \mathcal{P}_{fin}(\mathbb{N}) \to \{0, 1\}$ such that for any set $\mathcal{A}$ with size $\leq k$ and any set $\mathcal{S}$ such that $\mathcal{S}$ generates an IP set, $\mathcal{A}$ does not full-match $\mathcal{S}$.*

Proof. Fix a sequence $\{\mathcal{A}_i, \mathcal{W}_i\}$ where each $\mathcal{A}_i$ is a set of size k and each $\mathcal{W}_i$ is a computably enumerable set, and such that whenever $\mathcal{A}$ is a set of size k and $\mathcal{W}_i$ is a computably enumerable set, there is an i with $\mathcal{A}_i = \mathcal{A}$ and $\mathcal{W}_i = \mathcal{W}$.

For each s and each $i \leq s$ and $u \in [0, k]$, we inductively define $W_{i,s}^u$ to be least satisfying the following properties:

- $i < \min W_{i,s}^u$
- If $j < i$ and $W_{j,s}^{u'}$ is defined then $\max W_{j,s}^{u'} < \min W_{i,s}^u$
- If $u' < u$ and $W_{i,s}^{u'}$ is defined then $\max W_{i,s}^{u'} < \min W_{i,s}^u$
- $W_{i,s}^u \in \mathcal{W}_{i,s}$

If there is no such $W_{i,s}^u$ then $W_{i,s}^u$ is undefined.

A *decomposition* of B with $\max B = s$ is a tuple i, u, Z, D such that $B = Z \cup W_{i,s}^u \cup D$ and neither Z nor D contains $W_{i,s}^{u'}$ for any $u' \neq u$. We often write that $Z \cup W_{i,s}^u \cup D$ is a decomposition of B to mean that the tiple i, u, z, D is. A decomposition is *correct* if $Z \in \mathcal{A}_i$. (Recall that when we write $Z \cup W_{i,s}^u \cup D$, we implicitly assume that $\max Z < \min W_{i,s}^u$ and $\max W_{i,s}^u < \min D$.)

We define the *blocked* decompositions of B by induction on the length of D. A decomposition $B = Z \cup W^u_{i,s} \cup D$ is *blocked* by i' if there is an unblocked decomposition $B = Z' \cup W^{u'}_{i',s} \cup D'$ with $Z' = Z \cup Z_0$, and a Z_1 not containing $W^{u_0}_{i,s}$ for any u_0 such that $(Z_1 \cup Z_0) \cup W^{u'}_{i',s} \cup D'$ is a correct decomposition of $Z_1 \cup W^u_{i,s} \cup D$.

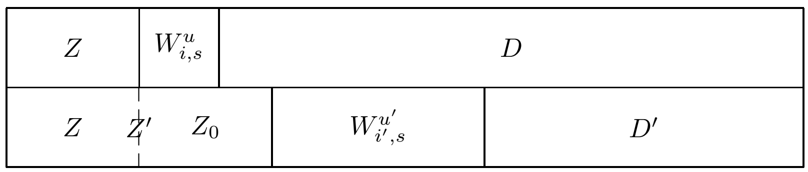

FIGURE 1. Two decompositions of the same set B

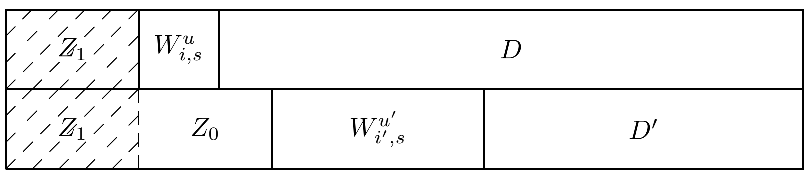

FIGURE 2. If $Z_1 \cup Z_0 \in \mathcal{A}_{i'}$ and $Z' \cup W^{u'}_{i',s} \cup D'$ is not blocked then the decomposition $Z \cup W^u_{i,s} \cup D$ is blocked.

Claim 1. *There is at most one correct unblocked decomposition.*

Proof. Suppose $B = Z \cup W^u_{i,s} \cup D = Z' \cup W^{u'}_{i',s} \cup D'$ give two correct decompositions, with Z a proper intial segment of Z'. Then by definition, $Z_1 = Z$ witnesses that $Z \cup W^u_{i,s} \cup D$ is blocked. ⊣

We now define our coloring inductively. Let B be given, and suppose $c(B)$ has been decided for all proper final segments of B. Let $Z \cup W^u_{i,s} \cup D$ be the correct, unblocked decomposition, if there is one. Then set $c(Z \cup W^u_{i,s} \cup D) = 1 - c(W^u_{i,s} \cup D)$. If there is no correct unblocked decomposition, set $c(B) = 0$.

Let a set $\mathcal{A}$ with $|\mathcal{A}| = k$ and a computably enumerable $\mathcal{W}$ be given. Chose n so that $\mathcal{W}_n = \mathcal{W}$ and $\mathcal{A}_n = \mathcal{A}$. If $\mathcal{W}_n$ generates an IP set, we may find W^u_n for $u \leq k$ and an s such that $W^u_{n,t} = W^u_n$ for all $t \geq s$, and find a $B \in \mathcal{W}_n$ with $\min B \geq s$.

Claim 2. *Suppose $A \cup W^v_n \cup B$ is blocked by i while $A' \cup W^{v'}_n \cup B$ is blocked by i'. Then $i = i'$.*

Proof. First, it follows from the definition that $A \cup W^v_n \cup B$ being blocked by i does not depend on the choice of A. So we may assume $A = A'$.

Suppose $i < i'$. Then we have $A \cup W^v_n \cup B = Z \cup W^u_i \cup D$ and $A \cup W^{v'}_n \cup B = Z' \cup W^{u'}_{i'} \cup D'$, with $Z = A \cup Z_0$ and $Z' = A \cup Z'_0$. Then there is a Z'_1 such that $(Z'_1 \cup Z'_0) \cup W^{u'}_{i'} \cup D'$ is a correct decomposition. Since $\max W^v_n < \min W^u_i$, there is a Z^* such that $Z^* \cup W^u_i \cup D = (Z'_1 \cup Z'_0) \cup W^{u'}_{i'} \cup D'$, and since $Z^* \setminus [0, \max W^v_n \cup W^{v'}_n] = Z \setminus [0, \max W^v_n \cup W^{v'}_n]$ and $\max W^v_n \cup W^{v'}_n < \min W^{u_0}_i$ for any u_0, $W^{u_0}_i$ does not occur in Z^*. Therefore therefore $Z \cup W^u_i \cup D$ is blocked (and therefore cannot itself block $A \cup W^v_n \cup B$). ⊣

It follows that there is at most one i such that $A \cup W_n^v \cup B$ is blocked by i for all $A \in \mathcal{A}_n, v \leq k$. In order for $A \cup W_n^v \cup B$ to be blocked by i, there must be a $Z \in \mathcal{A}_i$ such that W_n^v is contained in Z, and for $v \neq v'$, $W_n^{v'}$ is not contained in Z. Since $|\mathcal{A}_i| = k$, there are at most k values of v for which any $A \cup W_n^v \cup B$ is blocked. Therefore for some $v \leq k$, $A \cup W_n^v \cup B$ is a correct unblocked decomposition for all $A \in \mathcal{A}_n$, and therefore $c(A \cup W_n^v \cup B) = 1 - c(W_n^v \cup B)$. Since $W_n^v \cup B \in NU(\mathcal{W}_n)$, $\mathcal{A}_n$ does not half-match $NU(\mathcal{W}_n)$. □

3.3. **A Computable Coloring with No Computably Enumerable Full-Matched Sets.** Here we show that the first clause in Lemma 2.6 is necessary by presenting a computable coloring in which there is no finite set $\mathcal{B}$ and computable, or even computably enumerable, IP set $\mathcal{T}$ such that $\mathcal{B}$ full-matches $\mathcal{T}$.

Theorem 3.3. *There is a computable $c : \mathcal{P}_{fin}(\mathbb{N}) \to \{0, 1\}$ such that for any finite set $\mathcal{B}$ and any set $\mathcal{S}$ such that $\mathcal{S}$ generates an IP set, $\mathcal{B}$ does not full-match $\mathcal{S}$.*

Proof. For each s and each $i \leq s$ and $u \in \{0, 1\}$, we inductively define $W_{i,s}^u$ to be least satisfying the following properties:

- $i < \min W_{i,s}^u$
- If $j < i$ and $W_{j,s}^{u'}$ is defined then $\max W_{j,s}^{u'} < \min W_{i,s}^u$
- If $W_{i,s}^0$ is defined then $\max W_{i,s}^0 < \min W_{i,s}^1$
- $W_{i,s}^u \in \mathcal{W}_{i,s}$

If there is no such $W_{i,s}^u$ then $W_{i,s}^u$ is undefined.

A *primary s-decomposition* of B, where $s = \max B$, is a tuple i, u, Z, D such that $B = Z \cup W_{i,s}^u \cup D$, neither Z nor D contains $W_{i,s}^{1-u}$ as a subsequence, and there is no primary s-decomposition of D. Clearly there is at most one primary s-decomposition of B.

We say B *contains* i with polarity v if there is a primary $\max B$-decomposition j, u, Z, D of B with either $i = j$ and $v = u$, or i contained in Z with polarity $|v - u|$. Observe that whenever B contains i, $B = Z \cup W_{i,t}^u \cup D$ for some $t \leq \max B$.

We now define our coloring inductively. Let B be given, and suppose we have already decided $c(B')$ whenever B' is a proper initial segment of B. If B has a primary s-decomposition $B = Z \cup W_{i,s}^u \cup D$, we set $c(B) = c(Z)$ if $u = 0$ and $c(B) \neq c(Z)$ if $u = 1$. If there is no primary s-decomposition of B, we set $c(B) = 0$.

Claim 1. *For each i, there is some s such that $W_i^s = W_i^t$ for all $t \geq s$ (where both sides are undefined if either is).*

Let $\mathcal{B}$ be a finite set such that for all $A \in \mathcal{B}$, $\max A \leq i$ and let s, W_i^0, W_i^1 be such that for all $t \geq s$, $W_{i,s}^u = W_i^u$. It is easy to see that for any B with $\min B \geq s$ there is a v_B such that, $A \cup W_i^u \cup B$ contains i with polarity $|v_B - u|$ for all $A \in \mathcal{B}$.

Claim 2. *For all B with $\min B \geq s$, $c(A \cup W_i^{v_B} \cup B) = c(A)$ and $c(A \cup W_i^{1-v_B} \cup B) \neq c(A)$.*

Proof. By induction on the length of B. Let $D = A \cup W_i^u \cup B$. $A \cup W_i^u \cup B$ gives a primary $\max B$-decomposition of D unless B has a primary $\max B$-decomposition, so D must have a primary $\max B$-decomposition $Z \cup W_j^{u'} \cup B'$. If we just have $j = i$, the claim follows immediately from the definition of the coloring.

Otherwise, if $u' = 0$ then $c(D) = c(Z)$ and Z contains i with polarity $|v_B - u|$; by IH applied to $Z \setminus A \cup W_i^u$, $c(D) = c(Z) = c(A)$ if $u = v_B$ and $c(D) = c(Z) \neq c(A)$

if $u \neq v_B$. If $u' = 1$ then $c(D) \neq c(Z)$ and Z contains i with polarity $1 - |v_B - u|$; by IH applied to $Z \setminus A \cup W_i^u$, $c(D) \neq c(Z) \neq c(A)$ if $u = v_B$, so $c(D) = c(A)$, and $c(D) \neq c(Z) = c(A)$ if $u \neq v_B$. $\dashv$

So suppose $\mathcal{A}$ full-matched $NU(\mathcal{W})$ with $\mathcal{W}_e$ computably enumerable. Then for some i such that $\max A \leq i$ for all $A \in \mathcal{A}$, we have $\mathcal{W} = \mathcal{W}_i$. If $\mathcal{W}_i$ generated an IP set, there would be a $B \in \mathcal{W}_i$ with $\min B \geq s$, and $W_i^0, W_i^1 \in \mathcal{W}_i$ such that either A failed to full-match $W_i^0 \cup B$ or A failed to full-match $W_i^1 \cup B$. In either case, since both $W_i^0 \cup B$ and $W_i^1 \cup B$ belong to $NU(\mathcal{W}_i)$, $\mathcal{A}$ fails to full-match $NU(\mathcal{W}_i)$. $\square$

3.4. **A Computable Coloring with No Σ_2 Monochromatic IP Set.**

Theorem 3.4. *There is a computable $c : \mathcal{P}_{fin}(\mathbb{N}) \to \{0,1\}$ such that if $\mathcal{S}$ is a Σ_2 set generating an IP set then $NU(\mathcal{S})$ is not monochromatic.*

Proof. Fix an enumeration of all Σ_2 formulas

$$\phi_i(Z) = \exists x \forall y R_i(x, y, Z).$$

We will sometimes conflate ϕ_i with $\{Z \in \mathcal{P}_{fin}(\mathbb{N}) \mid \phi_i(Z)\}$ (for example, by writing $NU(\phi_i)$).

We arrange pairs (i, n) with $n < i + 1$ in lexicographic order (so $(j, m) < (i, n)$ iff $j < i$ or $j = i$ and $m < n$). For each pair (i, n), we define the (i, n)-candidates and $T_{i,n}$, the (i, n)-witness, simultaneously by induction.

Definition 3.5. *T is an i, n-candidate if:*

- $\phi_i(T)$
- *For each $(j, m) < (i, n)$ such that $T_{j,m}$ is defined, $\exists x \leq \min T_{i,n} \forall y R_j(x, y, T_{j,m})$*
- *For all $(j, m) < (i, n)$ such that $T_{j,m}$ is defined, $\max T_{j,m} < \min T_{i,n}$*

We define $T_{i,n}$, the i, n-witness, to be the least i, n-candidate if there is one, and undefined otherwise.

Note that if ϕ_i generates an IP set then all the i, n-witnesses are defined.

We will also need certain approximations to the i, n-witnesses.

Definition 3.6. *Let integers p, q be given. T is a p, q, i, n-candidate if:*

- $\exists x \leq p \forall y \leq q R_i(x, y, T_{i,n}^{p,q})$
- *For all $(j, m) < (i, n)$ such that $T_{j,m}^{p,q}$ is defined, $T_{j,m}^{\min T_{i,n}, q} = T_{j,m}^{p,q}$*
- *For all $(j, m) < (i, n)$ such that $T_{j,m}^{p,q}$ is defined, $\max T_{j,m}^{p,q} < \min T_{i,n}^{p,q}$*

We define $T_{i,n}^{p,q}$, the p, q, i, n-witness, to be the least p, q, i, n-candidate if there is one, and undefined otherwise.

Claim 1. *If $p \leq p'$ and $T_{j,m}^{p,q} = T_{j,m}^{p',q}$ for all $(j, m) < (i, n)$ then $T_{i,n}^{p',q} \preceq T_{i,n}^{p,q}$.*

Proof. It suffices to show that $T_{i,n}^{p,q}$ is a p', q, i, n-candidate. Certainly if $\exists x \leq p \forall y \leq q R_i(x, y, T_{i,n}^{p,q})$ then there is such an $x \leq p'$ as well. The remaining conditions hold by assumption. $\dashv$

Claim 2. *If $p \leq p' \leq p''$ and $T_{j,m}^{p,q} = T_{j,m}^{p'',q}$ for all $(j, m) < (i, n)$ then $T_{j,m}^{p',q} = T_{j,m}^{p,q}$ for all $(j, m) \leq (i, n)$.*

Proof. Suppose not. Let (j, m) be least such that $T_{j,m}^{p,q} \neq T_{j,m}^{p',q}$. Applying the preceeding lemma to p, p' and to p', p'', we have $T_{j,m}^{p',q} \prec T_{j,m}^{p,q} = T_{j,m}^{p'',q} \prec T_{j,m}^{p',q}$, which is impossible. $\dashv$

We define a coloring of $\mathcal{P}_{fin}(\mathbb{N})$ as follows. Let $B \in \mathcal{P}_{fin}(\mathbb{N})$ be given with $\max B = s$; we may assume $c(B')$ is decided for all B' with $\max B' < s$ and for all proper final segments of B. We will attempt to color B in a series of stages, indexed by $i \leq s$. At stage i, we ask whether there exist A, D such that:

- $A \cup D = B$,
- $\max A < \min D$, and
- $A = T_{i,n}^{\min D, \max D}$ for some $n < i+1$.

If all these conditions are met, we set $c(B) = 1 - c(D)$ for the longest such D, and say that B is i, A, D-colored. Otherwise, we do not color B at stage i.

If B is not colored at any stage $i \leq s$, we arbitrarily set $c(B) = 0$.

For each i, we wish to show that if ϕ_i generates an IP set then c is not monochromatic on $FU(\phi_i)$. So suppose ϕ_i generates an IP set. Choose p such that for each $T' \preceq T_{i,i}$, if $\exists x \forall y R_i(x,y,T')$ then $\exists x \leq p \forall y R_i(x,y,T')$. Since ϕ_i generates an IP set, we may find an A with $\min A \geq p$ and $\phi_i(A)$. Now let q be large enough that for each $j \leq i$, each $T' \preceq T_{i,i}$ such that $\neg\phi_j(T')$, and each $x \leq \min A$, there is a $y \leq q$ such that $\neg R_j(x,y,T')$. Again we may find B such that $\phi_i(B)$ and $\max B \geq q$. In particular, when $j \leq i$, $T' \preceq T_{i,i}$, $\exists x \forall y R_j(x,y,T')$ holds iff $\exists x \leq \min A \forall y \leq \max B R_j(x,y,T')$ holds, and therefore $T_{j,m} = T_{j,m}^{\min A, \max B}$ for all $(j,m) \leq (i,i)$.

We will show that for some $n < i+1$, $T_{i,n} \cup A \cup B$ is $i, T_{i,n}, A \cup B$-colored. This means $c(T_{i,n} \cup A \cup B) \neq c(A \cup B)$, and therefore $NU(\phi_i)$ is not monochromatic. Since $T_{i,n} = T_{i,n}^{\min A, \max B}$, it suffices to show that for some $n < i+1$, $T_{i,n} \cup A \cup B$ is not j, T', D-colored for any $j < i$ with $T' \neq T_{i,n}$ or i, T', D-colored for any T' a proper initial segment of $T'_{i,n}$.

Claim 3. *If T' is a proper initial segment of $T_{i,n}$ and $j < i$ then $T_{i,n} \cup A \cup B$ is not j, T', D-colored, where $T' \cup D = T_{i,n} \cup A \cup B$.*

Proof. Since $T_{j',m}^{\min T', \max D} = T_{j',m}^{\min A, \max D} = T_{j',m}$ for all $(j',m) < (i,n)$ and $\min T' \leq \min D \leq \min A$, it follows that $T_{j',m}^{\min D, \max D} = T_{j',m}$ for all $(j',m) < (i,n)$. In particular, since T' is a proper initial segment of $T_{i,n}$, we cannot have $T' = T_{j,m}$ for any m. Therefore $T_{i,n} \cup A \cup B$ is not j, T', D-colored. ⊣

Claim 4. *If T' is a proper initial segment of $T_{i,n}$ and $j < i$ then $T_{i,n} \cup A \cup B$ is not i, T', D-colored, where $T' \cup D = T_{i,n} \cup A \cup B$.*

Proof. If $\phi_i(T')$ then T' would be an i,n-candidate with $T' \prec T_{i,n}$, contradicting leastness of $T_{i,n}$. So $\neg\phi_i(T')$, and therefore $\forall x \leq \min A \exists y \leq \max B \neg R_j(x,y,T')$. Since $\max B = \max D$ and $\min D \leq \min A$, also $\forall x \leq \min D \exists y \leq \max D \neg R_j(x,y,T')$, so T' cannot be $T_{i,m}^{\min D, \max D}$ for any m. ⊣

It is still possible for $T_{i,n} \cup A \cup B$ to be j, T', D-colored by some $j < i$ when T' is a proper end-extension of $T_{i,n}$. We will show that each j does so for at most one $n < i+1$.

Claim 5. *If $j < i$ and $T_{i,n} \cup A \cup B$ is j, T', D-colored where T' is a proper end-extension of $T_{i,n}$ then $T' = T_{j,m}^{\min D, \max D}$ is least such that $T_{j,m}$ is undefined.*

Proof. By definition, $T' = T_{j,m}^{\min D, \max D}$ for some $m < j+1$. If $T_{j,m'}$ is defined for some $m' < j+1$ then, since $\min A \leq \min D$, $T_{j,m'} = T_{j,m'}^{\min D, \max D} \neq T'$. If $m' < m$ is such that $T_{j,m'}$ is undefined, in order for $T' \cup D$ to be j, T', D-colored, we would have to have $T_{j,m'}^{\min T', \max D} = T_{j,m'}^{\min D, \max D}$. But $\min T' = \min T_{i,n}$ and $\max D = \max B$,

so $T_{j,m'}^{\min T',\max D}=T_{j,m'}^{\min T_{i,n},\max B}$ is undefined. Therefore m is least such that $T_{j,m'}$ is undefined. $\dashv$

So suppose there are distinct $n,n'<i+1$ such that $T_{i,n}\cup A\cup B$ is j,T',D'-colored while $T_{i,n'}\cup A\cup B$ is j,T'',D''-colored. Without loss of generality, assume $T'\prec T''$. Then $\max D'=\max D''=\max B$ and $\min A\leq\min D'\leq\min D''$. Let m be least such that $T_{j,m}$ is undefined. Then T' is a $\min D'',\max D'',j,m$-candidate. Since $T'\prec T''$, it follows that T'' cannot be $T_{j,m}^{\min D'',\max D''}$.

Therefore for each j, there is at most one n such that $T_{i,n}\cup A\cup B$ is j,T',D-colored. This means there are at most i choices of n such that $T_{i,n}\cup A\cup B$ is j,T',D-colored for any $j<i$, and since there are $i+1$ possible values for n, there is some n such that $T_{i,n}\cup A\cup B$ is not j,T',D-colored for any $j<i$, and therefore $T_{i,n}\cup A\cup B$ is $i,T_{i,n},A\cup B$-colored, as desired. $\square$

4. Conclusion

The results of the previous section still leave a significant gap in the strength of Hindman's Theorem; in particular, while we do not see how to prove Hindman's Theorem if $\mathbf{ACA_0}$, we cannot rule out the possibility that there is such a proof.

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